

Multitime hybrid differential games with curvilinear integral functional

Constantin Udriște, Elena-Laura Otobîcu, Ionel Țevy

Multitime differential games are related to the modeling and analysis of cooperation or conflict in the context of a multitime dynamical systems. Their theory involves either a curvilinear integral functional or a multiple integral functional and an m -flow as constraint. The aim of this paper is to give original results regarding multitime hybrid differential games with curvilinear integral functional constrained by an m -flow: fundamental properties of multitime upper and lower values, viscosity solutions of multitime (HJIU) PDEs, representation formula of viscosity solutions for multitime (HJ) PDEs, and max-min representations.

Mathematics Subject Classification 2010: 49L20, 91A23, 49L25, 35F21.

Key words: multitime hybrid differential games, curvilinear integral cost, multitime dynamic programming, multitime viscosity solutions.

1 Multitime hybrid differential game with curvilinear integral functional

Let $t = (t^\alpha) \in \Omega_{0T} \subset \mathbb{R}_+^m$, $\alpha = 1, \dots, m$, be an evolution multi-parameter, called multitime. Consider an arbitrary C^1 curve Γ_{0T} joining the diagonal opposite points $0 = (0, \dots, 0)$ and $T = (T^1, \dots, T^m)$ in the m -dimensional parallelepiped $\Omega_{0T} = [0, T]$ (multitime interval) in \mathbb{R}_+^m endowed with the product order, a C^2 state vector $x : \Omega_{0T} \rightarrow \mathbb{R}^n$, $x(t) = (x^i(t))$, $i = 1, \dots, n$, a C^1 control vector $u(t) = (u_\alpha(t)) : \Omega_{0T} \rightarrow U \subset \mathbb{R}^{qm}$, for the first equip of m players (who wants to maximize), a C^1 control vector $v(t) = (v_\alpha(t)) : \Omega_{0T} \rightarrow V \subset \mathbb{R}^{qm}$, for the second equip of m players (who wants to minimize), $u_\alpha(\cdot) = \Phi(\cdot, \eta_1(\cdot))$, $v_\alpha(\cdot) = \Psi(\cdot, \eta_2(\cdot))$, a running cost $L_\alpha(t, x(t), u_\alpha(t), v_\alpha(t))dt^\alpha$ as a

nonautonomous closed Lagrangian 1-form (satisfies $D_\beta L_\alpha = D_\alpha L_\beta$), a terminal cost $g(x(T))$ and the C^1 vector fields $X_\alpha = (X_\alpha^i)$ satisfying the complete integrability conditions (CIC) $D_\beta X_\alpha = D_\alpha X_\beta$ (m-flow type problem).

In our paper, a *multitime hybrid differential game* is given by a multitime dynamics, as a PDE system controlled by two controllers (first equip, second equip) and a target including a curvilinear integral functional. The approach we follow below is those in the paper [2], but we must be more creative since our theory is multitemporal one (see also [8]-[20]). More precisely, we introduce and analyze a multitime differential game whose Bolza payoff is the sum between a path independent curvilinear integral (mechanical work) and a function of the final event (the terminal cost, penalty term), and whose evolution PDE is an m-flow: *Find*

$$\min_{v(\cdot) \in V} \max_{u(\cdot) \in U} J(u(\cdot), v(\cdot)) = \int_{\Gamma_{0T}} L_\alpha(s, x(s), u_\alpha(s), v_\alpha(s)) ds^\alpha + g(x(T)),$$

subject to the Cauchy problem

$$\frac{\partial x^i}{\partial s^\alpha}(s) = X_\alpha^i(s, x(s), u_\alpha(s), v_\alpha(s)),$$

$$x(0) = x_0, \quad s \in \Omega_{0T} \subset \mathbb{R}_+^m, \quad x \in \mathbb{R}^n.$$

Let D_α be the total derivative operator and $[X_\alpha, X_\beta]$ be the bracket of vector fields. Suppose the piecewise complete integrability conditions (CIC)

$$\left(\frac{\partial X_\alpha}{\partial u_\lambda^a} \delta_\beta^\gamma - \frac{\partial X_\beta}{\partial u_\lambda^a} \delta_\alpha^\gamma \right) \frac{\partial u_\lambda^a}{\partial s^\gamma} + \left(\frac{\partial X_\alpha}{\partial v_\lambda^b} \delta_\beta^\gamma - \frac{\partial X_\beta}{\partial v_\lambda^b} \delta_\alpha^\gamma \right) \frac{\partial v_\lambda^b}{\partial s^\gamma} = [X_\alpha, X_\beta] + \frac{\partial X_\beta}{\partial s^\alpha} - \frac{\partial X_\alpha}{\partial s^\beta},$$

where $a, b = 1, \dots, q$, are satisfied throughout.

To simplify, suppose that the curve Γ_{0T} is an increasing curve in the multitime interval Ω_{0T} . If we vary the starting multitime and the initial point, then we obtain a larger family of similar multitime problems containing the functional

$$J_{x,t}(u(\cdot), v(\cdot)) = \int_{\Gamma_{tT}} L_\alpha(s, x(s), u_\alpha(s), v_\alpha(s)) ds^\alpha + g(x(T)),$$

and the evolution constraint

$$\frac{\partial x^i}{\partial s^\alpha}(s) = X_\alpha^i(s, x(s), u_\alpha(s), v_\alpha(s)),$$

$$x(t) = x, \quad s \in \Omega_{tT} \subset \mathbb{R}_+^m, \quad x \in \mathbb{R}^n.$$

We assume that each vector field $X_\alpha : \Omega_{0T} \times \mathbb{R}^n \times U \times V \rightarrow \mathbb{R}^n$ is uniformly continuous, satisfying

$$\begin{cases} \|X_\alpha(t, x, u_\alpha, v_\alpha)\| \leq A_\alpha \\ \|X_\alpha(t, x, u_\alpha, v_\alpha) - X_\alpha(t, \hat{x}, u_\alpha, v_\alpha)\| \leq A_\alpha \|x - \hat{x}\|, \end{cases}$$

for some constant 1-form $A = (A_\alpha)$ and all $t \in \Omega_{0T}$, $x, \hat{x} \in \mathbb{R}^n$, $u \in U$, $v \in V$.

Suppose the functions

$$g : \mathbb{R}^n \rightarrow \mathbb{R}, \quad L_\alpha : \Omega_{0T} \times \mathbb{R}^n \times U \times V \rightarrow \mathbb{R}$$

are uniformly continuous and satisfy the boundedness conditions

$$\begin{cases} |g(x)| \leq B \\ |g(x) - g(\hat{x})| \leq B \|x - \hat{x}\|, \end{cases}$$

$$\begin{cases} |L_\alpha(t, x, u_\alpha, v_\alpha)| \leq C_\alpha \\ |L_\alpha(t, x, u_\alpha, v_\alpha) - L_\alpha(t, \hat{x}, u_\alpha, v_\alpha)| \leq C_\alpha \|x - \hat{x}\|, \end{cases}$$

for constant 1-form $C = (C_\alpha)$ and all $t \in \Omega_{0T}$, $x, \hat{x} \in \mathbb{R}^n$, $u \in U$, $v \in V$.

Definition 1.1. (i) The set

$$\mathcal{U}(t) = \{u_\alpha(\cdot) : \mathbb{R}_+^m \rightarrow U \mid u_\alpha(\cdot) \text{ is measurable and satisfies CIC}\}$$

is called **the control set for the first equip of players**. (ii) The set

$$\mathcal{V}(t) = \{v_\alpha(\cdot) : \mathbb{R}_+^m \rightarrow V \mid v_\alpha(\cdot) \text{ is measurable and satisfies CIC}\}$$

is called **the control set for the second equip of players**.

Definition 1.2. (i) A map $\Phi : \mathcal{V}(t) \rightarrow \mathcal{U}(t)$ is called **a strategy for the first equip of players**, if the equality $v(\tau) = \widehat{v}(\tau)$, $t \leq \tau \leq s \leq T$ implies $\Phi[v](\tau) = \Phi[\widehat{v}](\tau)$. (ii) A map $\Psi : \mathcal{U}(t) \rightarrow \mathcal{V}(t)$ is called **a strategy for the second equip of players**, if the equality $u(\tau) = \widehat{u}(\tau)$, $t \leq \tau \leq s \leq T$ implies $\Psi[u](\tau) = \Psi[\widehat{u}](\tau)$.

Let $\mathcal{A}(t)$ be the set of strategies for the first equip of players and $\mathcal{B}(t)$ be the set of strategies for the second equip of players.

Definition 1.3. (i) The function

$$m(t, x) = \min_{\Psi \in \mathcal{B}} \max_{u(\cdot) \in U} J_{t,x}(u(\cdot), \Psi[u](\cdot))$$

is called **the multitime lower value function**. (ii) The function

$$M(t, x) = \max_{\Phi \in \mathcal{A}} \min_{v(\cdot) \in V} J_{t,x}(\Phi[v](\cdot), v(\cdot))$$

is called **the multitime upper value function**.

The multitime lower value function $m(t, x)$ and the multitime upper value function $M(t, x)$ are piecewise continuously differentiable (see below, the boundedness and continuity of the values functions).

2 Properties of lower and upper values

Theorem 2.1. (multitime dynamic programming optimality conditions) For each pair of strategies (Φ, Ψ) , the lower and upper value functions can be written respectively in the form

$$m(t, x) = \min_{\Psi \in \mathcal{B}(t)} \max_{u_\alpha \in \mathcal{U}(t)} \left\{ \int_{\Gamma_{tt+h}} L_\alpha(s, x(s), u_\alpha(s), \Psi[u_\alpha](s)) ds^\alpha + m(t+h, x(t+h)) \right\} \quad (2.1)$$

and

$$M(t, x) = \max_{\Phi \in \mathcal{A}(t)} \min_{v_\alpha \in \mathcal{V}(t)} \left\{ \int_{\Gamma_{tt+h}} L_\alpha(s, x(s), \Phi[v_\alpha](s), v_\alpha(s)) ds^\alpha + M(t+h, x(t+h)) \right\}, \quad (2.2)$$

for all $(t, x) \in \Omega_{tT} \times \mathbb{R}^n$ and all $h \in \Omega_{0T-t}$.

Proof. First we recognize the Bellman principle (we write the value of a decision problem at a certain point in multitime in terms of the payoff from some initial choices and the value of the remaining decision problem that results from those initial choices).

To confirm the first statement, we shall use the function

$$w(t, x) = \min_{\Psi \in \mathcal{B}(t)} \max_{u_\alpha \in \mathcal{U}(t)} \left\{ \int_{\Gamma_{tt+h}} L_\alpha(s, x(s), u_\alpha(s), \Psi[u_\alpha](s)) ds^\alpha + m(t+h, x(t+h)) \right\}. \quad (2.3)$$

We will show that, for all $\varepsilon > 0$, the lower value function $m(t, x)$ will satisfies two inequalities, $m(t, x) \leq w(t, x) + 2\varepsilon$ and $m(t, x) \geq w(t, x) - 3\varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows $m(t, x) = w(t, x)$.

i) For $\varepsilon > 0$, there exists a strategy $\Upsilon \in \mathcal{B}(t)$ such that

$$w(t, x) \geq \max_{u_\alpha \in \mathcal{U}(t)} \left\{ \int_{\Gamma_{tt+h}} L_\alpha(s, x(s), u_\alpha(s), \Upsilon[u_\alpha](s)) ds^\alpha + m(t+h, x(t+h)) \right\} - \varepsilon. \quad (2.4)$$

We shall use the state $x(\cdot)$ which solves the (PDE), with the initial condition $\bar{x} = x(t+h)$ (Cauchy problem) on the set $\Omega_{tT} \setminus \Omega_{tt+h}$, for each $\bar{x} \in \mathbb{R}^n$. We can write

$$m(t+h, \bar{x}) = \min_{\Psi \in \mathcal{B}(t+h)} \max_{u_\alpha \in \mathcal{U}(t+h)} \left\{ \int_{\Gamma_{t+hT}} L_\alpha(s, x(s), u_\alpha(s), \Psi[u_\alpha](s)) ds^\alpha + g(x(T)) \right\}. \quad (2.5)$$

Thus there exists a strategy $\Upsilon_{\bar{x}} \in \mathcal{B}(t+h)$ for which

$$m(t+h, \bar{x}) \geq \max_{u_\alpha \in \mathcal{U}(t+h)} \left\{ \int_{\Gamma_{t+hT}} L_\alpha(s, x(s), u_\alpha(s), \Upsilon_{\bar{x}}[u_\alpha](s)) ds^\alpha + g(x(T)) \right\} - \varepsilon. \quad (2.6)$$

Define a new strategy

$$\Psi \in \mathcal{B}(t), \Psi[u_\alpha](s) \equiv \begin{cases} \Upsilon[u_\alpha](s) & s \in \Omega_{tt+h} \\ \Upsilon_{\bar{x}}[u_\alpha](s) & s \in \Omega_{tT} \setminus \Omega_{tt+h}, \end{cases}$$

for each control $u_\alpha \in \mathcal{U}(t)$. For any $u_\alpha \in \mathcal{U}(t)$, replacing the inequality (2.6) in the inequality (2.4), we obtain

$$w(t, x) \geq \int_{\Gamma_{tT}} L_\alpha(s, x(s), u_\alpha(s), \Psi[u_\alpha](s)) ds^\alpha + g(x(T)) - 2\varepsilon.$$

Consequently

$$\max_{u_\alpha \in \mathcal{U}(t)} \left\{ \int_{\Gamma_{tT}} L_\alpha(s, x(s), u_\alpha(s), \Psi[u_\alpha](s)) ds^\alpha + g(x(T)) \right\} \leq w(t, x) + 2\varepsilon.$$

Hence

$$m(t, x) \leq w(t, x) + 2\varepsilon.$$

- ii) On the other hand, there exists a strategy $\Psi \in \mathcal{B}(t)$ for which we can write the inequality

$$m(t, x) \geq \max_{u_\alpha \in \mathcal{U}(t)} \left\{ \int_{\Gamma_{tT}} L_\alpha(s, x(s), u_\alpha(s), \Psi[u_\alpha](s)) ds^\alpha + g(x(T)) \right\} - \varepsilon. \quad (2.7)$$

By the definition of $w(t, x)$, we have

$$w(t, x) \leq \max_{u_\alpha \in \mathcal{U}(t)} \left\{ \int_{\Gamma_{tt+h}} L_\alpha(s, x(s), u_\alpha(s), \Psi[u_\alpha](s)) ds^\alpha + m(t+h, x(t+h)) \right\} \quad (2.8)$$

and consequently there exists a control $u_\alpha^1 \in \mathcal{U}(t)$ such that

$$w(t, x) \leq \int_{\Gamma_{tt+h}} L_\alpha(s, x(s), u_\alpha^1(s), \Psi[u_\alpha^1](s)) ds^\alpha + m(t+h, x(t+h)) + \varepsilon. \quad (2.9)$$

Define a new control

$$u_\alpha^* \in \mathcal{U}(t), u_\alpha^*(s) \equiv \begin{cases} u_\alpha^1(s) & s \in \Omega_{tt+h} \\ u_\alpha(s) & s \in \Omega_{tT} \setminus \Omega_{tt+h}, \end{cases}$$

for each control $u_\alpha \in \mathcal{U}(t+h)$ and then define the strategy $\Psi^* \in \mathcal{B}(t+h)$, $\Psi^*[u_\alpha](s) \equiv \Psi[u_\alpha^*](s)$, $s \in \Omega_{tT} \setminus \Omega_{tt+h}$. We find the inequality

$$\begin{aligned} & m(t+h, x(t+h)) \\ & \leq \max_{u_\alpha \in \mathcal{U}(t+h)} \left\{ \int_{\Gamma_{tt+h}} L_\alpha(s, x(s), u_\alpha(s), \Psi^*[u_\alpha](s)) ds^\alpha + g(x(T)) \right\} \end{aligned} \quad (2.10)$$

and so there exists the control $u_\alpha^2 \in \mathcal{U}(t+h)$ for which

$$\begin{aligned} & m(t+h, x(t+h)) \\ & \leq \int_{\Gamma_{tT} \setminus \Gamma_{tt+h}} L_\alpha(s, x(s), u_\alpha^2(s), \Psi^*[u_\alpha^2](s)) ds^\alpha + g(x(T)) + \varepsilon. \end{aligned} \quad (2.11)$$

Define a new control

$$u_\alpha \in \mathcal{U}(t), u_\alpha(s) \equiv \begin{cases} u_\alpha^1(s) & s \in \Omega_{tt+h} \\ u_\alpha^2(s) & s \in \Omega_{tT} \setminus \Omega_{tt+h}. \end{cases}$$

Then the inequalities (2.9) and (2.11) yield

$$w(t, x) \leq \int_{\Gamma_{tT}} L_\alpha(s, x(s), u_\alpha(s), \Psi[u_\alpha](s)) ds^\alpha + g(x(T)) + 2\varepsilon,$$

and so (2.7) implies the inequality

$$w(t, x) \leq m(t, x) + 3\varepsilon.$$

This inequality and $m(t, x) \leq w(x, t) + 2\varepsilon$ complete the proof.

□

Theorem 2.2. (boundedness and continuity of the values functions) *The lower, upper value function $m(t, x)$, $M(t, x)$ satisfy the boundedness conditions*

$$|m(t, x)|, |M(t, x)| \leq D$$

$$|m(t, x) - m(\hat{t}, \hat{x})|, |M(t, x) - M(\hat{t}, \hat{x})| \leq E \ell(\Gamma_{\hat{t}t}) + F \|x - \hat{x}\|,$$

for some constants D, E, F and for all $t, \hat{t} \in \Omega_{0T}$, $x, \hat{x} \in \mathbb{R}^n$.

Proof. We prove only the statements for upper value function $M(t, x)$.

Since $|g(x)| \leq B$, $|L_\alpha(t, x, u_\alpha, v_\alpha)| \leq C_\alpha$, $\alpha = \overline{1, m}$, we find

$$\begin{aligned} |J_{t,x}(u(\cdot), v(\cdot))| &= \left| \int_{\Gamma_{tT}} L_\alpha(s, x(s), u_\alpha(s), v_\alpha(s)) ds^\alpha + g(x(T)) \right| \\ &\leq \left| \int_{\Gamma_{tT}} L_\alpha(s, x(s), u_\alpha(s), v_\alpha(s)) ds^\alpha \right| + |g(x(T))| \\ &\leq \int_{\Gamma_{tT}} \|L_\alpha(s, x(s), u_\alpha(s), v_\alpha(s))\| \|ds^\alpha\| + |g(x(T))| \\ &\leq \|C\| \int_{\Gamma_{tT}} ds + B = \|C\| l(\Gamma_{tT}) + B \leq \|C\| l(\Gamma_{0T}) + B = D \\ &\implies |M(t, x)| \leq D, \end{aligned} \tag{2.12}$$

for all $u_\alpha(\cdot) \in \mathcal{U}(t)$, $v_\alpha(\cdot) \in \mathcal{V}(t)$.

Let $x_1, x_2 \in \mathbb{R}^n$, $t_1, t_2 \in \Omega_{0T}$. For $\varepsilon > 0$ and the strategy $\Phi \in \mathcal{A}(t_1)$, we have

$$M(t_1, x_1) \leq \min_{v_\alpha \in \mathcal{V}(t_1)} J(\Phi[v_\alpha], v_\alpha) + \varepsilon. \tag{2.13}$$

Define the control

$$\bar{v}_\alpha \in \mathcal{V}(t_1), \bar{v}_\alpha(s) \equiv \begin{cases} v_\alpha^1(s) & s \in \Omega_{0t_2} \setminus \Omega_{0t_1} \\ v_\alpha(s) & s \in \Omega_{0T} \setminus \Omega_{0t_2}, \end{cases}$$

for any $v_\alpha \in \mathcal{V}(t_2)$ and some $v_\alpha^1 \in V$ and for each $v_\alpha \in \mathcal{V}(t_2)$, $\underline{\Phi} \in \mathcal{A}(t_2)$ (the restriction of Φ over $\Omega_{0T} \setminus \Omega_{0t_1}$) by $\underline{\Phi}[v_\alpha] = \Phi[\bar{v}_\alpha]$, $s \in \Omega_{0T} \setminus \Omega_{0t_2}$.

Choose the control $v_\alpha \in \mathcal{V}(t_2)$ so that

$$M(t_2, x_2) \geq J(\underline{\Phi}[v_\alpha], v_\alpha) - \varepsilon. \tag{2.14}$$

By the inequality (2.13), we have

$$M(t_1, x_1) \leq J(\Phi[\bar{v}_\alpha], \bar{v}_\alpha) + \varepsilon. \tag{2.15}$$

We know that the (unique, Lipschitz) solution $x(\cdot)$ of the Cauchy problem

$$\begin{cases} \frac{\partial x^i}{\partial s^\alpha}(s) = X_\alpha^i(s, x(s), u_\alpha(s), v_\alpha(s)) \\ x(t) = x, \quad s \in \Omega_{tT} \subset \mathbb{R}_+^m, x \in \mathbb{R}^n, i = \overline{1, n}, \alpha = \overline{1, m}, \end{cases}$$

is the response to the controls $u_\alpha(\cdot), v_\alpha(\cdot)$ for $s \in \Omega_{0T}$.

We choose $x_1(\cdot)$ as solution of the Cauchy problem

$$\begin{cases} \frac{\partial x_1^i}{\partial s^\alpha}(s) = X_\alpha^i(s, x_1(s), \Phi[\bar{v}_\alpha](s), \bar{v}_\alpha(s)) \\ x_1(t_1) = x_1, \quad s \in \Omega_{0T} \setminus \Omega_{0t_1}. \end{cases}$$

Equivalently, $x_1(\cdot)$ is solution of integral equation

$$x_1(s) = x_1(t_1) + \int_{\Gamma_{t_1 s}} X_\alpha(\sigma, x_1(\sigma), \Phi[\bar{v}_\alpha](\sigma), \bar{v}_\alpha(\sigma)) d\sigma^\alpha.$$

Take $x_2(\cdot)$ as solution of the Cauchy problem

$$\begin{cases} \frac{\partial x_2^i}{\partial s^\alpha}(s) = X_\alpha^i(s, x_2(s), \underline{\Phi}[v_\alpha](s), v_\alpha(s)) \\ x_2(t_2) = x_2, \quad s \in \Omega_{0T} \setminus \Omega_{0t_2}. \end{cases}$$

Equivalently, $x_2(\cdot)$ is solution of integral equation

$$x_2(s) = x_2(t_2) + \int_{\Gamma_{t_2 s}} X_\alpha(\sigma, x_2(\sigma), \underline{\Phi}[v_\alpha](\sigma), \bar{v}_\alpha(\sigma)) d\sigma^\alpha.$$

It follows that

$$\|x_1(t_2) - x_1\| = \|x_1(t_2) - x_1(t_1)\| \leq \|A\| \ell(\Gamma_{t_1 t_2}).$$

Since $v_\alpha = \bar{v}_\alpha$ and $\underline{\Phi}[v_\alpha] = \Phi[\bar{v}_\alpha]$, for $s \in \Omega_{0T} \setminus \Omega_{0t_2}$, we find the estimation

$$\begin{aligned} \|x_1(s) - x_2(s)\| &\leq \|x_1(t_1) - x_2(t_2)\| + \left\| \int_{\Gamma_{t_1 t_2}} \cdots \right\| \\ &\leq \|A\| \ell(\Gamma_{t_1 t_2}) + \|x_1 - x_2\|, \text{ on } t_2 \leq s \leq T. \end{aligned} \tag{2.16}$$

Thus the inequalities (2.14) and (2.15) imply

$$\begin{aligned} M(t_1, x_1) - M(t_2, x_2) &\leq J(\Phi[\bar{v}_\alpha], \bar{v}_\alpha) - J(\underline{\Phi}[v_\alpha], v_\alpha) + 2\varepsilon \\ &\leq \left| \int_{\Gamma_{t_1 t_2}} L_\alpha(s, x_1(s), \Phi[\bar{v}_\alpha](s), \bar{v}_\alpha(s)) ds^\alpha \right| \end{aligned}$$

$$\begin{aligned}
& + \int_{\Gamma_{t_2 T}} (L_\alpha(s, x_1(s), \underline{\Phi}[v_\alpha](s), v_\alpha(s)) - L_\alpha(s, x_2(s), \underline{\Phi}[v_\alpha](s), v_\alpha(s))) ds^\alpha \\
& \quad + |g(x_1(T)) - g(x_2(T)) + 2\varepsilon| \\
& \leq \int_{\Gamma_{t_1 t_2}} |L_\alpha(s, x_1(s), \Phi[\bar{v}_\alpha](s), \bar{v}_\alpha(s)) ds^\alpha| \\
& + \int_{\Gamma_{t_2 T}} |(L_\alpha(s, x_1(s), \underline{\Phi}[v_\alpha](s), v_\alpha(s)) - L_\alpha(s, x_2(s), \underline{\Phi}[v_\alpha](s), v_\alpha(s))) ds^\alpha| \\
& \quad + |g(x_1(T)) - g(x_2(T))| + 2\varepsilon \\
& \leq \|C\| \ell(\Gamma_{t_1 t_2}) + \|C\| \ell(\Gamma_{t_2 T}) (\|A\| \ell(\Gamma_{t_1 t_2}) + \|x_1 - x_2\|) + B \|x_1 - x_2\| + 2\varepsilon \\
& \leq \|C\| \ell(\Gamma_{t_1 t_2}) + \|C\| \ell(\Gamma_{0T}) (\|A\| \ell(\Gamma_{t_1 t_2}) + \|x_1 - x_2\|) + B \|x_1 - x_2\| + 2\varepsilon.
\end{aligned}$$

Since ε is arbitrary, we obtain the inequality

$$M(t_1, x_1) - M(t_2, x_2) \leq E \ell(\Gamma_{t_1 t_2}) + F \|x_1 - x_2\|. \quad (2.17)$$

Let $\varepsilon > 0$ and choose the strategy $\Phi \in \mathcal{A}(t_2)$ such that

$$M(t_2, x_2) \leq \min_{v_\alpha \in \mathcal{V}(t_2)} J(\Phi[v_\alpha], v_\alpha) + \varepsilon. \quad (2.18)$$

For each control $v_\alpha \in \mathcal{V}(t_1)$ and $s \in \Omega_{0T} \setminus \Omega_{0t_2}$, define the control $\underline{v}_\alpha \in \mathcal{V}(t_2)$, $\underline{v}_\alpha(s) = v_\alpha(s)$.

For some $u_\alpha^1 \in U$, we define the strategy $\bar{\Phi} \in \mathcal{A}(t_1)$ (the restriction of Φ over $\Omega_{0T} \setminus \Omega_{0t_2}$) by

$$\bar{\Phi}[v_\alpha] = \begin{cases} u_\alpha^1 & s \in \Omega_{0t_2} \setminus \Omega_{0t_1} \\ \Phi[\underline{v}_\alpha] & s \in \Omega_{0T} \setminus \Omega_{0t_2}. \end{cases}$$

Now choose a control $v_\alpha \in \mathcal{V}(t_1)$ so that

$$M(t_1, x_1) \geq J(\bar{\Phi}[v_\alpha], v_\alpha) - \varepsilon. \quad (2.19)$$

By the inequality (2.18), we have

$$M(t_2, x_2) \leq J(\Phi[\underline{v}_\alpha], \underline{v}_\alpha) + \varepsilon. \quad (2.20)$$

We choose $x_1(\cdot)$ as solution of the Cauchy problem (PDE system + initial condition)

$$\begin{cases} \frac{\partial x_1^i}{\partial s^\alpha}(s) = X_\alpha^i(s, x_1(s), \bar{\Phi}[v_\alpha], v_\alpha(s)), & s \in \Omega_{0T} \setminus \Omega_{0t_1} \\ x_1(t_1) = x_1, & s \in \Omega_{0T} \setminus \Omega_{0t_1} \end{cases}$$

and $x_2(\cdot)$ as solution of the Cauchy problem (PDE system + initial condition)

$$\begin{cases} \frac{\partial x_2^i}{\partial s^\alpha}(s) = X_\alpha^i(s, x_2(s), \Phi[\underline{v}_\alpha], \underline{v}_\alpha(s)), s \in \Omega_{0T} \setminus \Omega_{0t_2} \\ x_2(t_2) = x_2, \quad s \in \Omega_{0T} \setminus \Omega_{0t_2}. \end{cases}$$

Using the associated integral equations, it follows that

$$\|x_1(t_2) - x_1\| = \|x_1(t_2) - x_1(t_1)\| \leq \|A\| \ell(\Gamma_{t_1 t_2}).$$

Also, for $s \in \Omega_{0T} \setminus \Omega_{0t_2}$, $v_\alpha = \underline{v}_\alpha$ and $\bar{\Phi}[v_\alpha] = \Phi[\underline{v}_\alpha]$, we find

$$\begin{aligned} \|x_1(s) - x_2(s)\| &\leq \|x_1(t_1) - x_2(t_2)\| + \left\| \int_{\Gamma_{t_1 t_2}} \cdots \right\| \\ &\leq \|A\| \ell(\Gamma_{t_1 t_2}) + \|x_1 - x_2\|, \text{ on } t_2 \leq s \leq T. \end{aligned} \quad (2.21)$$

Thus, the relations (2.19) and (2.20) imply

$$\begin{aligned} M(t_2, x_2) - M(t_1, x_1) &= J(\bar{\Phi}[v_\alpha], v_\alpha) - J(\Phi[\underline{v}_\alpha], \underline{v}_\alpha) + 2\varepsilon \\ &= - \int_{\Gamma_{t_1 t_2}} L_\alpha(s, x_1(s), \bar{\Phi}[v_\alpha](s), v_\alpha(s)) ds^\alpha \\ &\quad + \int_{\Gamma_{t_2 T}} (L_\alpha(s, x_1(s), \Phi[\underline{v}_\alpha](s), \underline{v}_\alpha(s)) - L_\alpha(s, x_2(s), \Phi[\underline{v}_\alpha](s), \underline{v}_\alpha(s))) ds^\alpha \\ &\quad + g(x_1(T)) - g(x_2(T)) + 2\varepsilon \\ &\leq \|C\| \ell(\Gamma_{t_1 t_2}) + \|C\| \ell(\Gamma_{0T}) (\|A\| \ell(\Gamma_{t_1 t_2}) + \|x_1 - x_2\|) + B \|x_1 - x_2\| + 2\varepsilon. \end{aligned}$$

Since ε is arbitrary, we obtain the inequality

$$M(t_2, x_2) - M(t_1, x_1) \leq E \ell(\Gamma_{t_1 t_2}) + F \|x_1 - x_2\|. \quad (2.22)$$

By 2.17 and 2.22, we proved the continuity of the lower and upper value functions. \square

3 Viscosity solutions of multitime (HJIU) PDEs

Theorem 3.1. *(PDEs for multitime upper value function, resp. multitime lower value function)*

The multitime upper value function $M(t, x)$ and the multitime lower value function $m(t, x)$ are the viscosity solutions of Hamilton-Jacobi-Isaacs-Udriște (HJIU) PDEs:

- the multitime upper (HJIU) PDEs

$$\frac{\partial M}{\partial t^\alpha}(t, x) + \min_{v_\alpha \in \mathcal{V}} \max_{u_\alpha \in \mathcal{U}} \left\{ \frac{\partial M}{\partial x^i}(t, x) X_\alpha^i(t, x, u_\alpha, v_\alpha) + L_\alpha(t, x, u_\alpha, v_\alpha) \right\} = 0,$$

with the terminal condition $M(T, x) = g(x)$,

- the multitime lower (HJIU) PDEs

$$\frac{\partial m}{\partial t^\alpha}(t, x) + \max_{u_\alpha \in \mathcal{U}} \min_{v_\alpha \in \mathcal{V}} \left\{ \frac{\partial m}{\partial x^i}(t, x) X_\alpha^i(t, x, u_\alpha, v_\alpha) + L_\alpha(t, x, u_\alpha, v_\alpha) \right\} = 0,$$

with the terminal condition $m(T, x) = g(x)$.

Remark 3.2. If we introduce the so-called upper and lower Hamiltonian 1-forms defined respectively by

$$H_\alpha^+(t, x, p) = \min_{v_\alpha \in \mathcal{V}} \max_{u_\alpha \in \mathcal{U}} \{p_i(t) X_\alpha^i(t, x, u_\alpha, v_\alpha) + L_\alpha(t, x, u_\alpha, v_\alpha)\},$$

$$H_\alpha^-(t, x, p) = \max_{u_\alpha \in \mathcal{U}} \min_{v_\alpha \in \mathcal{V}} \{p_i(t) X_\alpha^i(t, x, u_\alpha, v_\alpha) + L_\alpha(t, x, u_\alpha, v_\alpha)\},$$

then the multitime (HJIU) PDE systems can be written in the form

$$\frac{\partial M}{\partial t^\alpha}(t, x) + H_\alpha^+ \left(t, x, \frac{\partial M}{\partial x}(t, x) \right) = 0$$

and

$$\frac{\partial m}{\partial t^\alpha}(t, x) + H_\alpha^- \left(t, x, \frac{\partial m}{\partial x}(t, x) \right) = 0.$$

The proof will be given in another paper.

4 Representation formula of viscosity solutions for multitime (HJ) PDEs

In this section, we want to obtain a representation formula for the viscosity solution $M(t, x)$ of the multitime (HJ) PDEs system

$$\frac{\partial M}{\partial t^\alpha} + H_\alpha \left(t, x, \frac{\partial M}{\partial x}(t, x) \right) = 0, (t, x) \in \Omega_{0T} \times \mathbb{R}^n, \alpha = \overline{1, m}, \quad (4.1)$$

$$M(0, x) = g(x), x \in \mathbb{R}^n \text{ (initial condition)}, \quad (4.2)$$

where the unique solution $M(t, x)$ satisfies the inequalities

$$\begin{cases} |M(t, x)| \leq D \\ |M(t, x) - M(\hat{t}, \hat{x})| \leq E \ell(\Gamma_{t\hat{t}}) + F \|x - \hat{x}\|, \end{cases} \quad (4.3)$$

for some constants D, E, F (for $m = 1$, see also [4]).

Also, we assume that $g : \mathbb{R}^n \rightarrow \mathbb{R}, H_\alpha : \Omega_{0T} \times \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}$, satisfy the inequalities

$$\begin{cases} |g(x)| \leq B \\ |g(x) - g(\hat{x})| \leq B \|x - \hat{x}\| \end{cases}$$

and

$$\begin{cases} |H_\alpha(t, x, 0)| \leq K_\alpha \\ |H_\alpha(t, x, p) - H_\alpha(\hat{t}, \hat{x}, \hat{p})| \leq K_\alpha (\ell(\Gamma_{t\hat{t}}) + \|x - \hat{x}\| + \|p - \hat{p}\|). \end{cases} \quad (4.4)$$

Max-min representation of a Lipschitz function as affine functions (for $m = 1$, see also [2], [3]).

Lemma 4.1. *For each α , let*

$$\begin{cases} U = B(0, 1) \subset \mathbb{R}^n \\ V = B(0, P) \subset \mathbb{R}^n \\ X_\alpha(u_\alpha) = K_\alpha u_\alpha, K_\alpha \in \mathbb{R} \\ L_\alpha(t, x, u_\alpha, v_\alpha) = H_\alpha(t, x, v_\alpha) - \langle K_\alpha u_\alpha, v_\alpha \rangle. \end{cases} \quad (4.5)$$

Let H_α be a Lipschitz 1-form. For some constant $P > 0$ and for each $t \in \Omega_{0T}$, $x \in \mathbb{R}^n$, we have

$$H_\alpha(t, x, p) = \max_{v_\alpha \in V} \min_{u_\alpha \in U} \{ \langle X_\alpha(u_\alpha), p \rangle + L_\alpha(t, x, u_\alpha, v_\alpha) \},$$

if $\|p\| \leq P$.

Proof. In view of the assumption $H_\alpha(t, x, v_\alpha) - H_\alpha(t, x, p) \leq K_\alpha \|p - v_\alpha\|$, by the Cauchy-Schwarz formula, and by the condition $\|u\| \leq 1$, we have for any $x \in \mathbb{R}^n$,

$$\begin{aligned} H_\alpha(t, x, p) &= \max_{v_\alpha \in V} \{ H_\alpha(t, x, v_\alpha) - K_\alpha \|p - v_\alpha\| \} \\ &= \max_{v_\alpha \in V} \min_{u_\alpha \in U} \{ H_\alpha(t, x, v_\alpha) + \langle K_\alpha u_\alpha, p - v_\alpha \rangle \}. \end{aligned} \quad (4.6)$$

□

Max-min representation of a Lipschitz function as positive homogeneous functions (for $m=1$, see also [2],[3]).

Lemma 4.2. Let H_α be a Lipschitz 1-form which is homogeneous in p , i.e.,

$$H_\alpha(t, x, \lambda p) = \lambda H_\alpha(t, x, p), \quad \lambda \geq 0.$$

Then there exist compact sets $U, V \subset \mathbb{R}^{2n}$ and vector fields

$$X_\alpha : [0, T] \times \mathbb{R}^n \times U \times V \rightarrow \mathbb{R}^n$$

satisfying

$$\|X_\alpha(x) - X_\alpha(\hat{x})\| \leq A_\alpha \|x - \hat{x}\|$$

and such that, for each α ,

$$H_\alpha(t, x, p) = \max_{v_\alpha \in V} \min_{u_\alpha \in U} \{ \langle X_\alpha(t, x, u_\alpha, v_\alpha), p \rangle \},$$

for all $t \in \Omega_{0T}$, $x \in \mathbb{R}^n$, $p \in \mathbb{R}^n$.

Proof. Let $u_\alpha = (u_\alpha^1, u_\alpha^2)$, $v_\alpha = (v_\alpha^1, v_\alpha^2)$ ($2n$ -dimensional controls) and

$$\begin{cases} U = V = B(0, 1) \times B(0, 1) \subset \mathbb{R}^{2n} \\ L_\alpha(t, x, u_\alpha^1, v_\alpha^1) = H_\alpha(t, x, v_\alpha^1) - \langle K_\alpha u_\alpha^1, v_\alpha^1 \rangle \\ X_\alpha(t, x, u_\alpha, v_\alpha) = K_\alpha u_\alpha^1 + C_\alpha v_\alpha^2 + (L_\alpha(t, x, u_\alpha^1, v_\alpha^1) - C_\alpha) u_\alpha^2. \end{cases} \quad (4.7)$$

According to Lemma (4.1) and the assumptions (4.7), if $\|\eta\| = 1$, we have

$$H_\alpha(t, x, \eta) = \max_{v_\alpha^1 \in V^1} \min_{u_\alpha^1 \in U^1} \{ \langle K_\alpha u_\alpha^1, \eta \rangle + L_\alpha(t, x, u_\alpha^1, v_\alpha^1) \}, \quad (4.8)$$

for $U^1 = V^1 = B(0, 1) \in \mathbb{R}^n$.

For any $p \neq 0$, we can write

$$\begin{aligned} H_\alpha(t, x, p) &= \|p\| H_\alpha\left(t, x, \frac{p}{\|p\|}\right) \\ &= \max_{v_\alpha^1 \in V^1} \min_{u_\alpha^1 \in U^1} \{ \langle K_\alpha u_\alpha^1, p \rangle + L_\alpha(t, x, u_\alpha^1, v_\alpha^1) \|p\| \}. \end{aligned} \quad (4.9)$$

Then, if we choose $C_\alpha > 0$ such that $|L_\alpha| \leq C_\alpha$, we find

$$\begin{aligned} H_\alpha(t, x, p) &= \max_{v_\alpha^1 \in V^1} \min_{u_\alpha^1 \in U^1} \left\{ \langle K_\alpha u_\alpha^1, p \rangle + C_\alpha \|p\| + (L_\alpha(t, x, u_\alpha^1, v_\alpha^1) - C_\alpha) \|p\| \right\} \\ &= \max_{v_\alpha^1 \in V^1} \min_{u_\alpha^1 \in U^1} \max_{v_\alpha^2 \in V^1} \min_{u_\alpha^2 \in U^1} \left\{ \langle K_\alpha u_\alpha^1, p \rangle + \langle C_\alpha v_\alpha^2, p \rangle \right. \\ &\quad \left. + (L_\alpha(t, x, u_\alpha^1, v_\alpha^1) - C_\alpha) \langle u_\alpha^2, p \rangle \right\} \\ &= \max_{v_\alpha \in V} \min_{u_\alpha \in U} \left\{ \langle X_\alpha(t, x, u_\alpha, v_\alpha), p \rangle \right\}. \end{aligned} \quad (4.10)$$

Now, interchanging $\min_{u_\alpha^1 \in U^1}$ and $\max_{v_\alpha^2 \in V^1}$, the result in Lemma follows. \square

We are now in a position to give the main result of this section.

Theorem 4.3. *For each $t \in \Omega_{0T}$ and $x \in \mathbb{R}^n$, the upper value function $M(t, x)$ verifies the equality*

$$M(t, x) = \max_{\Phi \in \mathcal{U}(T-t)} \min_{v_\alpha \in V(T-t)} \left\{ - \int_{\Gamma_{T-t}} L_\alpha(T-s, x(s), \Phi[v_\alpha](s), v_\alpha(s)) ds^\alpha + g(x(T)) \right\}, \quad (4.11)$$

where for each pair of controls $v_\alpha \in V(T-t)$, $u_\alpha = \Phi[v_\alpha] \in U(T-t)$, the state function $x(\cdot)$ solves the problem

$$\begin{cases} \frac{\partial x^i}{\partial s^\alpha}(s) = -F_\alpha^i u_\alpha(s), s \in \Omega_{0T} \setminus \Omega_{0T-t} \\ x(T-t) = x. \end{cases} \quad (4.12)$$

Proof. Let

$$H_\alpha^1(t, x, p) = \max_{v_\alpha \in V} \min_{u_\alpha \in U} \{ \langle X_\alpha(u_\alpha), p \rangle + L_\alpha(t, x, u_\alpha, v_\alpha) \},$$

$U = B(0, 1) \subset \mathbb{R}^{pm}$, $V = B(0, P) \subset \mathbb{R}^{qm}$ and X_α^i, L_α Lipschitz functions with the assumptions (4.5).

Then $H_\alpha(t, x, p) = H_\alpha^1(t, x, p)$ provided $|p| \leq P$. Since $M(t, x)$ satisfies (4.3), it follows that $M(t, x)$ is also the unique viscosity solution of the multi-time (HJ) PDEs system (for $m = 1$, see also [4])

$$\frac{\partial M}{\partial t^\alpha} + H_\alpha^1 \left(t, x, \frac{\partial M}{\partial x}(t, x) \right) = 0, \quad (t, x) \in \Omega_{0T} \times \mathbb{R}^n, \alpha = \overline{1, m}, \quad (4.13)$$

$$M(0, x) = g(x), \quad x \in \mathbb{R}^n. \quad (4.14)$$

If we take $M^1(t, x) = M(T-t, x)$, one observes that $M^1(t, x)$ is a viscosity solution of this system (for $m = 1$, see also [2])

$$\frac{\partial M^1}{\partial t^\alpha} + H_\alpha^+ \left(t, x, \frac{\partial M^1}{\partial x}(t, x) \right) = 0, \quad (t, x) \in \Omega_{0T} \times \mathbb{R}^n, \alpha = \overline{1, m}, \quad (4.15)$$

$$M^1(T, x) = g(x), \quad x \in \mathbb{R}^n \quad (4.16)$$

and

$$H_\alpha^+(t, x, p) = \max_{v_\alpha \in V} \min_{u_\alpha \in U} \{ - \langle X_\alpha(u_\alpha), p \rangle + L_\alpha(T-t, x, u_\alpha, v_\alpha) \}.$$

Using the above developments, we obtain

$$M^1(t, x) = M(t, x) = \max_{\Phi \in \mathcal{U}(t)} \min_{v_\alpha \in V(t)} \left\{ - \int_{\Gamma_{tT}} L_\alpha(T - s, x(s), \Phi[v_\alpha](s), v_\alpha(s)) ds^\alpha + g(x(T)) \right\}, \quad (4.17)$$

where $x(\cdot)$ is the solution of the Cauchy problem

$$\begin{cases} \frac{\partial x^i}{\partial s^\alpha}(s) = -X_\alpha^i(u_\alpha(s)) = -F_\alpha^i u_\alpha(s), s \in \Omega_{0T} \setminus \Omega_{0T-t} \\ x(t) = x, \end{cases} \quad (4.18)$$

for the control $u_\alpha(\cdot) = \Phi[v_\alpha]$. □

References

- [1] L. C. Evans, *An Introduction to Mathematical Optimal Control Theory*, Lectures Notes, University of California, Departament of Mathematics, Berkeley, (2005).
- [2] L. C. Evans, P. E. Souganidis, *Differential games and representation formulas for solutions of Hamilton-Jacobi-Isaacs equations*, Indiana University Mathematics Journal, 33, 5, (1984), 773-797.
- [3] M. G. Crandall, L. C. Evans, P. L. Lions, *Some properties of viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc., 282, 2, (1984), 487-502.
- [4] M. G. Crandall, P. L. Lions, *Viscosity Solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc., 277, (1983), 1-42.
- [5] E. N. Barron, L. C. Evans, R. Jensen, *Viscosity solutions of Isaacs' equations and differential games with Lipschitz controls*, Journal of Differential Equations, 53, (1984), 213-233.
- [6] L. Gómez Esparza, G. Mendoza Torres, L. M. Saynes Torres, *A Brief Introduction to Differential Games*, International Journal of Physical and Mathematical Sciences, 4, 1, (2013).
- [7] G. Jank, *Introduction to Non-cooperative Dynamical Game Theory*, Coimbra, (2001).

- [8] P. E. Souganidis, *Existence of viscosity solutions of Hamilton-Jacobi equations*, Journal of Differential Equations, 56, (1985), 345-390.
- [9] K. Margellos, J. Lygeros, *Hamilton-Jacobi formulation for reach-avoid differential games*, IEEE Trans. Automat. Contr., 56, 8, (2011), 1849-1861.
- [10] C. Udriște, I. Țevy, *Multi-time Euler-Lagrange-Hamilton theory*, WSEAS Trans. Math., 6, 6, (2007), 701-709.
- [11] C. Udriște, *Multitime stochastic control theory*, in Selected Topics on Circuits, Systems, Electronics, Control and Signal Processing, Proc. of the 6-th WSEAS International Conference on Circuits, Systems, Electronics, Control and Signal Processing (CSECS07), Cairo, Egypt, December 29-31, (2007), 171-176.
- [12] C. Udriște, *Multi-time controllability, observability and bang-bang principle*, J. Optim. Theory Appl., 138, 1 (2008), 141-157.
- [13] C. Udriște, L. Matei, I. Duca, *Multitime Hamilton-Jacobi Theory*, Proceedings of the 8th WSEAS International Conference on Applied Computer and Applied Computational Science, 509-513, 2009.
- [14] C. Udriște, *Equivalence of multitime optimal control problems*, Balkan J. Geom. Appl. 15, 1, (2010), 155-162.
- [15] C. Udriște, *Simplified multitime maximum principle*, Balkan J. Geom. Appl. 14, 1, (2009), 102-119.
- [16] C. Udriște, I. Țevy, *Multitime dynamic programming for curvilinear integral actions*, J. Optim. Theory and Appl., 146, (2010), 189-207.
- [17] C. Udriște, L. Matei, *Lagrange-Hamilton Theories* (in Romanian), Monographs and Textbooks 8, Geometry Balkan Press, Bucharest, (2008).
- [18] C. Udriște, A. Bejenaru, *Multitime optimal control with area integral costs on boundary*, Balkan J. Geom. Appl., 16, 2, (2011), 138-154
- [19] C. Udriște, *Multitime maximum principle for curvilinear integral cost*, Balkan J. Geom. Appl., 16, 1, (2011), 128-149.
- [20] C. Udriște, I. Țevy, *Multitime dynamic programming for multiple integral actions*, J. Glob. Optim., 51, 2, (2011), 345-360.

- [21] A. W. Starr, *Nonzero-sum differential games: concepts and models*, Division of Engineering and Applied Physics Harvard University-Cambridge, Massachusetts, Technical Report, 590, (1969).
- [22] A. Davini, M. Zavidovique, *On the (non) existence of viscosity solutions of multi-time Hamilton-Jacobi equations*, Preprint (2013), <http://www.math.jussieu.fr/~zavidovique/articles/NonCommutingNov2013.pdf>